Announcements

1) Exam Tuesday covering
4.3, 4.6, 4.7, 4.10, 5.7,
10.2, 10.5, 10.7

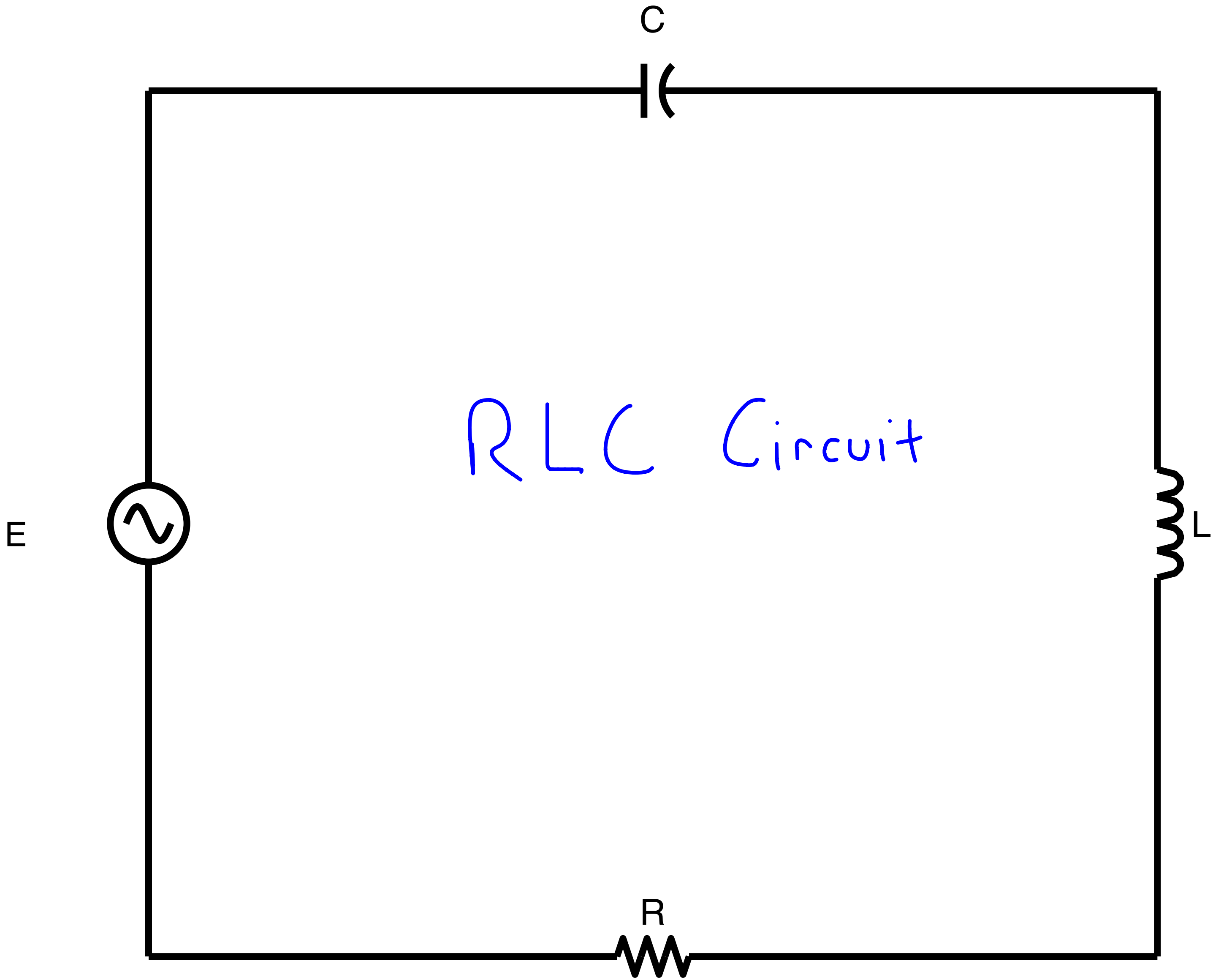
2) Practice problems on Canvas
under "Assignments"

RLC Circuits

(Section 5.7)

One resistor, one capacitor, one inductor

No systems of equations, just one equation!


C
RLC Circuit
E
L
R

New Equation

$$E_C + E_R + E_L = E(t)$$

$$E_R = RI$$

$$E_L = L\frac{dI}{dt}$$

$$E_C = \frac{q}{C} \quad \left(\frac{dq}{dt} = I(t)\right)$$

$$L\frac{dI}{dt} + RI + \frac{q}{c} = E(t)$$

Differentiate w.r.t. t

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{c}\frac{dq}{dt} = \frac{dE}{dt}$$

Substitute $\frac{dq}{dt} = I$

$$\boxed{L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{c}I = \frac{dE}{dt}}$$

Example 3: In a simple RLC circuit, if $R = 6\,\Omega$, $L = 1\,H$, $C = .2F$, $E(t) = \sin(30t)$, and the initial current and charge on the capacitor are zero, solve for $I(t)$.

$\hookrightarrow$ $I(0) = 0, \ \dfrac{dI}{dt}(0) = 0$

Plug into

$$L\frac{d^2I}{dt^2}+R\frac{dI}{dt}+\frac{1}{C}I=\frac{dE}{dt}$$

$$=30\cos(30t)$$

$$\frac{d^2I}{dt^2}+6\frac{dI}{dt}+\frac{1}{.2}I=30\cos(30t)$$

$$\boxed{\frac{d^2I}{dt^2}+6\frac{dI}{dt}+5I=30\cos(30t)}$$

Step 1: Solve homogeneous equation.

Constant coefficients, so

suppose in

$$\frac{d^2I}{dt^2} + 6\frac{dI}{dt} + 5I = 0,$$

let $I(t) = e^{rt}$. We get

$$e^{rt}(r^2 + 6r + 5) = 0, \text{ so}$$

$$\underbrace{r^2 + 6r + 5}_{(r+5)(r+1)} = 0$$

$$\boxed{r = -1, -5}$$

Solutions to homogeneous equation:

$$\boxed{y_1(t) = e^{-t}, \quad y_2(t) = e^{-5t}}$$

Step 2: Variation of Parameters!

Suppose $y_p(t) = u(t)y_1(t) + v(t)y_2(t)$.

We get

$$u'(t)y_1(t) + v'(t)y_2(t) = 0$$

$$u'(t)y_1'(t) + v'(t)y_2'(t) = 30\cos(30t)$$

$y_1(t) = e^{-t}, \quad y_1'(t) = -e^{-t}$

$y_2(t) = e^{-5t}, \quad y_2'(t) = -5e^{-5t}$

we get

$u'(t)e^{-t} + v'(t)e^{-5t} = 0$

$-u'(t)e^{-t} - 5v'(t)e^{-5t} = 30\cos(30t)$

Multiply 1st equation by

e^t.

$u'(t) + v'(t)e^{-4t} = 0$, so

$u'(t) = -v'(t)e^{-4t}$

Substitute into 2nd equation:

$$v'(t)e^{-5t} - 5v'(t)e^{-5t} = 30\cos(30t),$$

so

$$v'(t) = \frac{30e^{5t}\cos(30t)}{-4}$$

$$u'(t) = -v'(t)e^{-4t}$$

$$= \frac{30e^{t}\cos(30t)}{4}$$

Using Mathematica, we get

$$v(t) = -\frac{3}{74} e^{5t}\left(\cos(30t) + 6\sin(30t)\right)$$

$$u(t) = \frac{15}{1802} e^{t}\left(\cos(30t) + 30\sin(30t)\right)$$

and

$$y_p(t) = y_1(t)u(t) + y_2(t)v(t)$$

$$= \boxed{\frac{-6\left(179\cos(30t) - 36\sin(30t)\right)}{33{,}337}}$$

So

$$\boxed{\begin{aligned} I(t) &= y_p(t) + c_1 y_1(t) + c_2 y_2(t) \\ &= -\frac{6(179\cos(30t) - 36\sin(30t))}{33{,}337} \\ &\quad + c_1 e^{-t} + c_2 e^{-5t} \end{aligned}}$$

Now use intitial conditions:

$$I(0) = 0, \quad I'(0) = 0$$

$I(0)=0$

$$0=\frac{-6(179)}{33{,}337}+C_1+C_2$$

$$\boxed{C_2=-C_1+\frac{1074}{33{,}337}}$$

$\underline{I'(0)=0}$

$$I'(t) = -C_1 e^{-t} - 5C_2 e^{-5t}$$

$$+ 6\left(\frac{5370\sin(30t) + 1080\cos(30t)}{33337}\right)$$

$$I'(0) = -C_1 - 5C_2 + \frac{6(1080)}{33337}$$

$$= 0$$

$$0 = -C_1 - 5\left(\frac{1074}{33337} - C_1\right) + \frac{6480}{33337}$$

$$= 4C_1 + \frac{1110}{3337}$$

$$C_1 = \frac{-555}{2(33337)}$$

$$C_2 = -C_1 + \frac{1074}{33337}$$

$$= \frac{2703}{2(33337)}$$

$$\boxed{I(t) = \frac{-555}{2(33337)} e^{-t} + \frac{2703}{2(33337)} e^{-5t} - 6\left(\frac{179\cos(30t) - 36\sin(30t)}{33337} \right)}$$